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Universal Formulation for the N-Body Problem

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The universal formulation for the perturbed two-body problem is generalized to cover all gravitational N-body problems involving a dominant central mass. Its efficiency, when compared to conventional numerical integration, is shown in several examples. The convergence and numerical stability of the method is discussed, and a universal state transition matrix is obtained, which can be used either in a process of differential correction of an orbit or, as in the present case, to obtain an accurate estimation of global errors.

Nomenclature

c_k	= universal function
F, G, H	= Lagrangian functions
g_i	= angular momentum of m_i
k	= Gaussian gravitational constant
M	= mass of the central body
m_i	= mass of the ith body
$\boldsymbol{P}$	= perturbing acceleration
p	= order of the universal scheme
q	= regularized time
$\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i$	= position and velocity vectors of m_i
$\boldsymbol{r}_{i0}, \dot{\boldsymbol{r}}_{i0}$	= position and velocity vectors of m_i
r_i	= distance from m_i to M
$\boldsymbol{r}_{ij}, r_{ij}$	= vector and distance from m_i to m_j
$\boldsymbol{TR}_t$	= truncation error in the transfer $t_0 \to t$
t	= time
t_0	= initial time
z	= solution of the main equation
ϕ	= state transition matrix
τ_i	= intermediate time corresponding to m_i

I. Introduction

IN a previous paper[1] we have presented a universal formulation for the perturbed two-body problem, which yields an efficient method for computing an ephemerides from initial values of position and velocity.

Such a method involves a regularizing transformation of the independent variable t, for the case where the perturbing function may become close to a singularity. It also involves the introduction of the so-called universal functions, defined by the series

$$c_{k(\lambda^2)} = \sum_{i=0}^{\infty} \frac{(-\lambda^2)^i}{(2i+k)!} \tag{1}$$

which have a very fast convergence and some interesting properties that the reader may find described in Battin[2] and Stumpff[3,4] as applied to the unperturbed two-body problem.

In the present paper we show how the universal scheme can be applied to gravitational N-body problems involving a dominant central mass. In such a case, the equation of motion for the ith body relative to the central mass M is

$$\frac{d^2}{dt^2}\boldsymbol{r}_i + k^2(M+m_i)\frac{\boldsymbol{r}_i}{r_j^3} = k^2 \sum_{\substack{j=1\\ j\neq i}}^{N-1} m_j \left\{\frac{\boldsymbol{r}_{ji}}{r_{ji}^3} - \frac{\boldsymbol{r}_j}{r_j^3}\right\} \qquad i = 1, (N-1) \tag{2}$$

where $\boldsymbol{r}_{ji} = \boldsymbol{r}_j - \boldsymbol{r}_i$ and $r_{ji} = |\boldsymbol{r}_{ji}|$ the distance between bodies j and i. When the intermediate time τ_i for m_i, defined as

$$\tau_i = \sqrt{M+m_i}\,k(t-t_0) = \sqrt{M+m_i}\,\tau \tag{3}$$

is adopted as the independent variable, Eq. (2) can be written as

$$\frac{d^2}{d\tau_i^2}\boldsymbol{r}_i + \frac{\boldsymbol{r}_i}{r_i^3} = \boldsymbol{P}_i \tag{4}$$

The initial conditions of the problem are

$$\boldsymbol{r}_{i(t_0)} = \boldsymbol{r}_{i0} \qquad \dot{\boldsymbol{r}}_{i(t_0)} = \dot{\boldsymbol{r}}_{i0}; \qquad i = 1, N \tag{5}$$

and $\boldsymbol{P}_i$ denotes the perturbation on m_i resulting from the remaining $(N-2)$ bodies,

$$\boldsymbol{P}_i = \frac{1}{(M+m_i)} \sum_{\substack{j=1\\ j\neq i}}^{N-1} m_j \left\{\frac{\boldsymbol{r}_{ji}}{r_{ji}^3} - \frac{\boldsymbol{r}_j}{r_j^3}\right\} \tag{6}$$

But Eq. (4) is formally identical to the equation of motion in a perturbed two-body problem. Therefore, the universal formulation derived in Zadunaisky and Giordano[1] can be applied directly to obtain position and velocity of m_i at any instant t, by means of

$$\boldsymbol{r}_i = F_i\boldsymbol{r}_{i0} + G_i\dot{\boldsymbol{r}}_{i0} + H_i\boldsymbol{g}_{i0} \tag{7}$$

and

$$\dot{\boldsymbol{r}}_i = \dot{F}_i\boldsymbol{r}_{i0} + \dot{G}_i\dot{\boldsymbol{r}}_{i0} + \dot{H}_i\boldsymbol{g}_{i0} \tag{8}$$

where $\boldsymbol{g}_{i0} = \boldsymbol{r}_{i0} \times \dot{\boldsymbol{r}}_{i0}$ is the angular momentum vector orthogonal to the instantaneous plane of motion at the instant t_0 and $F_i, G_i, H_i, \dot{F}_i, \dot{G}_i$, and $\dot{H}_i$ are scalar functions of t, the components of $\boldsymbol{r}_{i0}, \dot{\boldsymbol{r}}_{i0}$ and the perturbations. Obviously, H_i and $\dot{H}_i$ become null when $\boldsymbol{P}_i$ is either null or belongs to the instantaneous plane of motion of the perturbed body. In such a case, formulas (7) and (8) are formally identical to the well-known Lagrangian formulation for the two-body problem.

The universal scheme is described in Appendix A, where the complete set of formulas is given, because they will help in the explanation of some theoretical results concerning the truncation

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errors of the method proposed in this paper, as well as other matters to be addressed in due time.

Here we also discuss the convergence and numerical stability of the universal method. For the sake of brevity, only the main features and conclusions of the analysis are offered, but a detailed discussion can be found in Giordano.[5]

Furthermore, the universal state transition matrix for the perturbed motion is obtained and an accurate method for the estimation of global truncation errors is provided.

Finally, in several examples we test the derived error estimates and show the proposed method can be applied to the case of the gravitational N-body problem and also to problems where the perturbations can be nongravitational and nonconservative, such as drag forces in artificial bodies.

II. Local Truncation Error: Statement of the Problem

To analyze the sources of truncation errors in the transfer $t_0 \to t$, let us examine the formulas given in Appendix A.

To compute $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ we apply formulas (A21) and (A22), respectively, for which it is necessary to previously solve the main equation (A15) and then to apply formulas (A23) to (A26).

When computing $\boldsymbol{r}$, however, the truncated expansions Φ^T, F^T, G^T, and H^T are used instead of the infinite series for Φ, F, G, and H. Then, if $\boldsymbol{r}^T$ denotes the truncated expression for $\boldsymbol{r}$ and z^* the solution of the truncated main equation, the local truncation error in the transfer $t_0 \to t$ can be written

$$\left[\boldsymbol{r}_{(z)} - \boldsymbol{r}^T_{(z^*)}\right] = \left[\boldsymbol{r}_{(z)} - \boldsymbol{r}^T_{(z)}\right] + \left[\boldsymbol{r}^T_{(z)} - \boldsymbol{r}^T_{(z^*)}\right] \tag{9}$$

Hence, the difference

$$\boldsymbol{r}^R_{(z)} = \left[\boldsymbol{r}_{(z)} - \boldsymbol{r}^T_{(z)}\right] \tag{10}$$

accounts for the residual series F^R, G^R, and H^R furnished by the truncation of F, G, and H, whereas

$$\left[\boldsymbol{r}^T_{(z)} - \boldsymbol{r}^T_{(z^*)}\right] \simeq -\dot{\boldsymbol{r}}^T_{(z^*)}\tau\Phi^R_{(z^*)} \tag{11}$$

accounts for the fact that not the main equation but the truncated main equation has been solved.

As regards the local truncation error in $\dot{\boldsymbol{r}}$, a similar analysis may be accomplished, yielding

$$\left[\dot{\boldsymbol{r}}_{(z)} - \dot{\boldsymbol{r}}^T_{(z^*)}\right] = \left[\dot{\boldsymbol{r}}_{(z)} - \dot{\boldsymbol{r}}^T_{(z)}\right] + \left[\dot{\boldsymbol{r}}^T_{(z)} - \dot{\boldsymbol{r}}^T_{(z^*)}\right] \tag{12}$$

where

$$\left[\dot{\boldsymbol{r}}_{(z)} - \dot{\boldsymbol{r}}^T_{(z)}\right] = \frac{\boldsymbol{r}^{(1)R}}{r_{(z)}} - \frac{r^R}{r r^T}\boldsymbol{r}^{(1)T}_{(Z)} \tag{13}$$

and

$$\left[\dot{\boldsymbol{r}}^T_{(z)} - \dot{\boldsymbol{r}}^T_{(z^*)}\right] \simeq \ddot{\boldsymbol{r}}^T_{(z^*)}\tau\Phi^R_{(z^*)} \tag{14}$$

Thus, we are faced with the need for asymptotic formulas for the residual series neglected in the computation to estimate the local truncation errors in the transfer $t_0 \to t$.

For that purpose we introduce in Sec. III a one-step method of integration, namely, a refinement of the Taylor series method, which will lead us not only to find the local truncation error estimates, but also to prove the stability and convergence of our method.

III. Modified Taylor Expansion

In Zadunaisky and Giordano[1] we have shown that, when Sundman's regularizing variable[6] is adopted as the independent variable, F satisfies

$$F^{(3)} + \alpha^2 F^{(1)} = 3r^3\sigma P_{r_0} + r^2 P^{(1)}_{r_0} \tag{15}$$

with $\alpha^2 = \rho r^2 - (\boldsymbol{r}\cdot\boldsymbol{P})$ [Eq. (47) in Ref. 1], and analogous differential equations are valid for both G and H.

In consequence, let us now consider the more general equation

$$u^{(s+2)}_{(q)} + \alpha^2 u^{(s)}_{(q)} = \varepsilon g_{(u,u^{(1)},\ldots,u^{(s+1)})} \qquad s \geq 0 \tag{16}$$

where $u_{(q)}$ is the unknown function, εg the perturbing term, ε a small parameter, and α^2 a constant coefficient. When ε is identically zero, Eq. (16) reduces to the unperturbed equation

$$u^{(s+2)}_{(q)} + \alpha^2 u^{(s)}_{(q)} = 0 \qquad s \geq 0 \tag{17}$$

The aim of this section is to derive a one-step method for integrating Eq. (16), which should have the property that if the perturbing term is switched off at an arbitrary instant of the independent variable q then the numerical method at hand should integrate without discretization error the subsequent unperturbed Eq. (17). This is a generalization of a method proposed by Stiefel and Scheifele[7] for the particular case of the perturbed harmonic oscillator [Eq. (16) with $s = 0$].

Defining the variables

$$y_i = u^{(i)} \qquad i = 0, (s+1) \tag{18}$$

Eq. (16) may be transformed into an equivalent first-order system

$$\boldsymbol{y}^{(1)} = \boldsymbol{f}_{(y,q)} \tag{19}$$

where $\boldsymbol{y}$ and $\boldsymbol{f}$ are the column vectors

$$\boldsymbol{y} = [\, y_0, \ldots, y_{s+1}]^t \tag{20a}$$

$$\boldsymbol{f} = [\, y_1, \ldots, y_{s+1}, -\alpha^2 y_s + \varepsilon g_{(y,q)}]^t \tag{20b}$$

where t is the transposing operator.

A well-known method for integrating Eq. (19) is the Taylor series method of order p, which can be written as

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\psi_{T(y_n,q_n;h)} \tag{21}$$

where

$$\psi_{T(y,q;h)} = \sum_{k=0}^{p-1} \frac{h^k}{(k+1)!}\boldsymbol{f}^{(k)}_{(y,q)} \tag{22}$$

is the increment function. Besides, $\boldsymbol{f}$ being $(p+1)$ times continuously differentiable, the local truncation error has the asymptotic form[8]

$$\boldsymbol{d}_{n(h)} = h^{p+1}\phi_{T(y,q)} + \mathbb{O}(h^{p+2}) \tag{23}$$

with the principal error function given by

$$\phi_T = -\frac{1}{(p+1)!}\frac{\mathrm{d}^p}{\mathrm{d}q^p}\boldsymbol{f}_{(y,q)} \tag{24}$$

Let us modify the increment function ψ_T by introducing Stumpff's functions[2-4]

$$c_{k(\lambda^2)} = \sum_{i=0}^{\infty} \frac{(-\lambda^2)^i}{(2i+k)!} \tag{25}$$

by means of their recurrence relation

$$1/k! = c_k + \lambda^2 c_{k+2} \tag{26}$$

with $\lambda^2 = \alpha^2 h^2$ being the value to be adopted for the argument. Then,

$$\psi_{T(y,q;h)} = \boldsymbol{f}^{(0)}c_1 + h\boldsymbol{f}^{(1)}c_2 + \sum_{k=2}^{p-1} h^k\left(\boldsymbol{f}^{(k)} + \alpha^2\boldsymbol{f}^{(k-2)}\right)c_{k+1}$$
$$+ \left(\alpha^2 h^p \boldsymbol{f}^{(p-2)}c_{p+1} + \alpha^2 h^{p+1}\boldsymbol{f}^{(p-1)}c_{p+2}\right) \tag{27}$$

and, by canceling the last two terms, a new increment function is obtained,

$$\psi_{s(y,q;h)} = \sum_{k=0}^{p-1} h^k \boldsymbol{b}_{k+1}c_{k+1} \tag{28}$$

where

$$\boldsymbol{b}_1 = \boldsymbol{f}, \qquad \boldsymbol{b}_2 = \boldsymbol{f}^{(1)}, \qquad \boldsymbol{b}_{k+1} = \boldsymbol{f}^{(k)} + \alpha^2\boldsymbol{f}^{(k-2)}, \qquad k \geq 2$$

We now define the Stumpff series method of order p by

$$y_{n+1} = y_n + h\psi_{s(y_n,q_n;h)} \tag{29}$$

and by applying the well-known general theorems on one-step methods,[7] we conclude that the Stumpff series method of order p is both stable and convergent provided f is p times continuously differentiable, and the local truncation error satisfies

$$d_{n(h)} = h^{p+1}\phi_{s(y,q)} + \mathbb{O}(h^{p+2}) \tag{30}$$

where the principal error function is given by

$$\phi_s = -\frac{1}{(p+1)!}\left\{f^{(p)}_{(y,q)} + \alpha^2 f^{(p-2)}_{(y,q)}\right\} \tag{31}$$

if f is $(p+1)$ times continuously differentiable.

The ith component of ϕ_S, namely,

$$\phi^i_s = -\frac{1}{(p+1)!}\epsilon g^{(p+i-s-1)}_{(y,q)} \qquad i = 0, (s+1), p \geq (s+1) \tag{32}$$

turns out to be proportional to ε. Therefore, the Stumpff series method integrates the unperturbed problem exactly, without discretization error, whereas the Taylor series method produces a truncation error even in the case where $\varepsilon = 0$ [see Eqs. (23) and (24)]. Moreover, the Stumpff series method provides a more precise numerical solution for Eq. (16) than that given by the Taylor series method of the same order.

IV. Asymptotic Expressions for the Truncation Error in the Transfer $t_0 \to t$

At this stage of the discussion it should be pointed out that the expressions derived for F, G, and H [Eqs. (50) in Ref. 1] are actually their Stumpff series expansions in terms of the regularized time.

Keeping this in mind, it can be easily concluded that

$$F^R = [F - F^T] = \frac{q^{p+1}}{(p+1)!}\left\{F_0^{(p+1)} + \alpha_0^2 F_0^{(p-1)}\right\} + \mathbb{O}(q^{p+2}) \tag{33a}$$

and

$$F^{(1)R} = [F^{(1)} - F^{(1)T}] = \frac{q^{p+1}}{(p+1)!}\left\{F_0^{(p+2)} + \alpha_0^2 F_0^{(p)}\right\} + \mathbb{O}(q^{p+2}) \tag{33b}$$

if F^T and $F^{(1)T}$ include up to the pth power of the regularized step q.

Equations (33a) and (33b) can be written in terms of $z = r_0 q/\tau$, the solution of the main equation. Then, it is

$$F^R = f_p\frac{(\tau z)^{p+1}}{(p+1)!} + \mathbb{O}(q^{p+2}) \tag{34a}$$

and

$$F^{(1)R} = f_{p+1}\frac{(\tau z)^{p+1}}{(p+1)!} + \mathbb{O}(q^{p+2}) \tag{34b}$$

since $f_k = \{F_0^{(k+1)} + \alpha_0^2 F_0^{(k-1)}\}/r_0^k$, whereas analogous estimates are valid for G^R, $G^{(1)R}$, H^R and $H^{(1)R}$ involving the coefficients g_p, g_{p+1}, h_p, and h_{p+1}, respectively.

Also the modulus r and the pseudotime τ, as functions of the regularized time, satisfy equations of the type Eq. (16) because

$$r^{(3)} + \alpha^2 r^{(1)} = r\left[3\left(\boldsymbol{r}^{(1)}\cdot\boldsymbol{P}\right) + \left(\boldsymbol{r}\cdot\boldsymbol{P}^{(1)}\right)\right] \tag{35}$$

[formula (20) in Ref. 1] and

$$\tau^{(1)} = r \tag{36}$$

Through a similar analysis we may arrive at the conclusion that the error because of the truncation of both r and the main equation turns out to be negligible compared to that introduced by the truncation of F, G, H, $F^{(1)}$, $G^{(1)}$, and $H^{(1)}$ series, which is certainly the main source of local error.

To sum up, the truncation error in the transfer $t_0 \to t$ can be estimated according to

$$\boldsymbol{TR}_t = \frac{(\tau z)^{p+1}}{(p+1)!}\begin{pmatrix} f_p x_0 + g_p\dot{x}_0 + h_p g_{0x} \\ f_p y_0 + g_p\dot{y}_0 + h_p g_{0y} \\ f_p z_0 + g_p\dot{z}_0 + h_p g_{0z} \\ (f_{p+1}/\Delta)x_0 + (g_{p+1}/\Delta)\dot{x}_0 + (h_{p+1}/\Delta)g_{0x} \\ (f_{p+1}/\Delta)y_0 + (g_{p+1}/\Delta)\dot{y}_0 + (h_{p+1}/\Delta)g_{0y} \\ (f_{p+1}/\Delta)z_0 + (g_{p+1}/\Delta)\dot{z}_0 + (h_{p+1}/\Delta)g_{0z} \end{pmatrix} + \mathbb{O}(q^{p+2}) \tag{37}$$

where $\boldsymbol{TR}_t$ is a six vector and $\Delta = r/r_0$. Numerical experiments yield results in close agreement with these conclusions (see Sec. VII).

V. Convergence and Stability

Note that $\boldsymbol{r}$ satisfies the regularized differential equation

$$\boldsymbol{r}^{(3)} + \alpha^2\boldsymbol{r}^{(1)} = 3r^3\sigma\boldsymbol{P} + r^2\boldsymbol{P}^{(1)} \tag{38}$$

which summarizes in vectorial form Eq. (15) for F and the analogous equations for G and H.

Furthermore, the expressions for $\boldsymbol{r}$ and $\boldsymbol{r}^{(1)}$ derived in the universal formulation are Stumpff series solutions of Eq. (38) up to a certain power of the integration step size in the independent variable; actually, they can be written as follows:

$$\boldsymbol{r}^T = F^T_{(q)}\boldsymbol{r}_0 + G^T_{(q)}\dot{\boldsymbol{r}}_0 + H^T_{(q)}\boldsymbol{g}_0 = \boldsymbol{r}_0 + q\psi^0_{s(\boldsymbol{r}_0,\boldsymbol{r}_0^{(1)},\boldsymbol{r}_0^{(2)};q)} \tag{39a}$$

and

$$\boldsymbol{r}^{(1)T} = F^{(1)T}_{(q)}\boldsymbol{r}_0 + G^{(1)T}_{(q)}\dot{\boldsymbol{r}}_0 + H^{(1)T}_{(q)}\boldsymbol{g}_0 = \boldsymbol{r}_0^{(1)} + q\psi^1_{s(\boldsymbol{r}_0,\boldsymbol{r}_0^{(1)},\boldsymbol{r}_0^{(2)};q)} \tag{39b}$$

where ψ_s^0 and ψ_s^1 are the first two components of the increment function ψ_s defined by Eq. (28) and $\lambda^2 = \alpha_0^2 q^2$ the argument to adopt in Stumpff's functions. The value of the regularized step q, corresponding to the requested time t, results from the main equation as a consequence of the analytical step regulation

$$\mathrm{d}\tau = r\,\mathrm{d}q \tag{40}$$

from which the main equation was derived.[1]

Therefore, from the results presented in Sec. III and the foregoing analysis we draw an important conclusion: the universal method of order p is both convergent and stable, provided the perturbation is p times continuously differentiable.

Besides, the second-order method integrates exactly, without discretization error, the unperturbed two-body problem. Actually, it yields the well-known solution for the uniform treatment of Keplerian motion.[3,4]

VI. Error Propagation: Universal State Transition Matrix for the Perturbed Problem

The state transition matrix

$$\phi = \begin{pmatrix} \dfrac{\partial\boldsymbol{r}}{\partial\boldsymbol{r}_0} & \dfrac{\partial\boldsymbol{r}}{\partial\dot{\boldsymbol{r}}_0} \\ \dfrac{\partial\dot{\boldsymbol{r}}}{\partial\boldsymbol{r}_0} & \dfrac{\partial\dot{\boldsymbol{r}}}{\partial\dot{\boldsymbol{r}}_0} \end{pmatrix} \tag{41}$$

linearly propagates state deviations in position and velocity in the following way:

$$\begin{pmatrix} \delta\boldsymbol{r} \\ \delta\dot{\boldsymbol{r}} \end{pmatrix} = \phi\begin{pmatrix} \delta\boldsymbol{r}_0 \\ \delta\dot{\boldsymbol{r}}_0 \end{pmatrix} \tag{42}$$

Table 1 Lagrangian problem: estimated error vs actual errors[a]

	$p=2$		$p=3$		$p=4$	
	Est err	Real err	Est err	Real err	Est err	Real err
Δx	+0.17D–07	+0.19D–07	–0.10D–09	–0.18D–09	–0.19D–12	–0.21D–12
Δy	+0.28D–07	+0.24D–07	+0.63D–10	+0.25D–09	–0.32D–12	–0.26D–12
$\Delta \dot{x}$	–0.24D–07	–0.18D–07	–0.69D–10	–0.29D–09	+0.27D–12	+0.20D–12
$\Delta \dot{y}$	+0.10D–07	+0.98D–08	+0.92D–10	–0.92D–09	–0.18D–12	–0.10D–12

[a]Units: [dist] = 1 astronomical unit = 1AU–[time] = 1 mean solar day = $1d$ – [mass] = 1 solar mass.

As an immediate consequence of

$$\begin{pmatrix} r \\ \dot{r} \end{pmatrix} = \begin{pmatrix} [F] & [G] & [H] \\ [\dot{F}] & [\dot{G}] & [\dot{H}] \end{pmatrix} \begin{pmatrix} r \\ \dot{r} \\ g_0 \end{pmatrix} \tag{43}$$

where $[F], \ldots, [\dot{H}]$ denote 3×3 diagonal matrices such as

$$[F] = \begin{pmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix} \tag{44}$$

the six columns of ϕ are given by

$$\frac{\partial}{\partial x_k}\begin{pmatrix} r \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \left[\frac{\partial F}{\partial x_k}\right] & \left[\frac{\partial G}{\partial x_k}\right] & \left[\frac{\partial H}{\partial x_k}\right] \\ \left[\frac{\partial \dot{F}}{\partial x_k}\right] & \left[\frac{\partial \dot{G}}{\partial x_k}\right] & \left[\frac{\partial \dot{H}}{\partial x_k}\right] \end{pmatrix} \begin{pmatrix} r_0 \\ \dot{r}_0 \\ g_0 \end{pmatrix} + \begin{pmatrix} [F] & [G] \\ [\dot{F}] & [\dot{G}] \end{pmatrix} e_k + \begin{pmatrix} [H] \\ [\dot{H}] \end{pmatrix} \frac{\partial}{\partial x_k} g_0 \tag{45}$$

where $x_k{:}x_0 \to \dot{z}_0$, $k{:}1 \to 6$; e_k is a six vector of components

$$e_{ki} = \delta_{ki} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad i = 1, 6 \tag{46}$$

and

$$\frac{\partial}{\partial x_k} g_0 = \left[\frac{\partial}{\partial x_k} g_{0x}, \frac{\partial}{\partial x_k} g_{0y}, \frac{\partial}{\partial x_k} g_{0z}\right]^t \tag{47}$$

Expressions for the partial derivatives $\partial F/\partial x_k, \ldots, \partial \dot{H}/\partial x_k, k = 1 \to 6$ valid to first-order are given in Appendix B.

VII. Numerical Examples

Estimation of errors

To check the derived error estimates, we considered the Lagrangian solution of the three-body problem. As is well known, the barycentric position of each body at an instant t is given by

$$R_{(t)} = \rho R_{(t_0)} \tag{48}$$

where ρ is the modulus of the radius vector of a virtual two-body problem characterized by given values of the eccentricity e and the pericenter distance q (Refs. 9 and 10).

At the initial time the three bodies are assumed to be located on the vertices of an equilateral triangle with unit sides. The three bodies are confined to a plane, and the triangular configuration is preserved, though the sides may vary periodically.

In the examples that follow we adopted the masses $m_1 = 1$, $m_2 = 4.5205283\text{D–}04$ and $m_3 = 0$, corresponding to the central, perturbing, and perturbed bodies, respectively, and a step of size of 1 day was chosen.

Errors in position and velocity, when applying the universal method of order p, were estimated according to

$$\begin{pmatrix} \delta r \\ \delta \dot{r} \end{pmatrix} = \phi \begin{pmatrix} \delta r_0 \\ \delta \dot{r}_0 \end{pmatrix} + TR_t \tag{49}$$

with the six-vector TR_t given by Eq. (37).

Table 2 Lagrangian problem: estimated errors vs actual errors[a]

	$e = 0.50$		$e = 0.90$	
	Est err	Real err	Est err	Real err
Δx	–0.27D–12	–0.21D–12	–0.10D–13	–0.78D–13
Δy	–0.24D–13	–0.23D–13	+0.41D–13	+0.32D–13
$\Delta \dot{x}$	–0.97D–13	–0.52D–13	–0.10D–13	–0.10D–13
$\Delta \dot{y}$	+0.31D–13	+0.31D–13	+0.12D–13	+0.15D–13

[a]Units: [dist] = 1AU – [time] = 1 mean solar day – [mass] = 1 solar mass.

Equation (49) provides a good estimate of global errors even though the terms involving the partial derivatives $(\partial F/\partial x_k), \ldots, (\partial \dot{H}/\partial x_k)$, $k = 1 \to 6$ were neglected in ϕ [see Eq. (45)].

Actual errors were obtained by direct comparison with the exact results of the Lagrangian analytical solution.

Example 1

We adopted the values $e = 0.15$ and $q = 1$ for the eccentricity and the pericenter distance of the virtual two-body probem. Table 1 shows estimated and actual errors in position and velocity after 200 steps when the universal method of order p was used, for different values of p.

The estimation of the accumulated errors was in all cases quite satisfactory even if the terms in the partial derivatives in ϕ were neglected. Table 1 also illustrates the gain in precision when a further term is taken up in the universal expansions.

Example 2

Table 2 displays results similar to those of Table 1 corresponding to the values $e = 0.5$ and 0.9 of the eccentricity of the virtual ellipse in the Lagrangian solution. It is well known that a large eccentricity results in a closer approach between the attracting bodies, in which case the regularization actually comes into play.

In Table 2 the universal method of order $p = 4$ with a stepsize $w = 1$ day was used. This example tests the performance of the error estimate for high eccentricities. Table 2 also makes evident that the universal method suffers no loss of stability or accuracy at high eccentricity, which has already been pointed out in Zadunaisky and Giordano.[1]

Application to N-Body Problem

As a simple illustration of the performance of the universal method when applied to the gravitational N-body problem, we computed an ephemerides for planets. The masses and initial conditions for $t_0 = JD2441200.5$ adopted for the computation are those given by Oesterwinter and Cohen.[11]

The integration of order 10 was also performed by the method of Everhart,[12] which has proven to be accurate and efficient as compared to other leading methods.

Table 3 shows the differences between the universal and Everhart's solutions at $JD2441600.5$ when a rather small step, namely, $0.1d$, was used in Everhart's[12] application.

The first column displays the accuracy achieved by the universal scheme of order $p = 4$ with a step size of 1 day.

The second column shows the gain in precision when a further term is taken up in the universal series (scheme of order $p = 5$) at nearly no extra cost in computing time. When doubling the step size (see the last column in Table 3) an accuracy comparable to that obtained by the scheme of order $p = 4$ was achieved but requiring half the computing time.

Therefore, to attain a desired precision it is advisable to increase the number of c_k functions taken up in the universal series rather than

Table 3 Solar system: difference between universal and Everhart's solutions at $t = JD2441600.5$[a]

CPU time IBM 4361	$p = 4$, $w = 1d$, 0^s:18.23		$p = 5$, $w = 1d$, 0^s:21.5		$p = 5$, $w = 2d$, 0^s:11.45	
	pos	vel	pos	vel	pos	vel
Mercury	1.D–09	1.D–09	1.D–11	1.D–11	1.D–10	1.D–10
Venus	1.D–10	1.D–10	1.D–12	1.D–12	1.D–11	1.D–11
E-M Bar	1.D–10	1.D–10	1.D–12	1.D–12	1.D–11	1.D–11
Mars	1.D–10	1.D–10	1.D–12	1.D–12	1.D–11	1.D–11
Jupiter	1.D–11	1.D–11	1.D–12	1.D–12	1.D–11	1.D–11
Saturn	1.D–11	1.D–11	1.D–12	1.D–12	1.D–11	1.D–11
Uranus	1.D–11	1.D–11	1.D–12	1.D–12	1.D–11	1.D–11
Neptune	1.D–11	1.D–11	1.D–12	1.D–12	1.D–11	1.D–11
Pluto	1.D–11	1.D–11	1.D–12	1.D–12	1.D–11	1.D–11

[a]Units: [dist] = 1AU – [time] = 1 mean solar day = $1d$ – [mass] = 1 solar mass.

Table 4 Artificial satellite: differences between universal and Everhart's solutions after one revolution[a]

CPU time IBM 4361	$p = 4$, 0^s:05.43		$p = 5$, 0^s:05.52	
	pos	vel	pos	vel
	1.D–14	1.D–13	1.D–16	1.D–15

[a]Units: [dist] = Eq. Rad. – [time] = min – [mass] = Earth mass.

Table 5 Artificial satellite: differences between universal and Everhart's solutions after 50 revolutions[a]

CPU time IBM 4361	$p = 4$, 2^s:31.3		$p = 5$, 2^s:45.7	
	pos	vel	pos	vel
	1.D–10	1.D–10	1.D–14	1.D–13

[a]Units: [dist] = Eq. Rad. – [time] = min – [mass] = Earth mass.

to shorten the step length. Of course, an increase in the number of admitted universal functions cannot compensate for an unreasonably large step size.

Nonconservative Case

A simple drag acting on an artificial satellite was considered. The satellite is initially at pericenter at 500 kms above the surface of Earth. The unperturbed Kepler orbit is nearly circular ($e = 0.0001$) and the inclination of the orbital plane with respect to the equator is 28 deg.

The drag acceleration was computed simply as

$$\boldsymbol{P} = -\beta\rho V \boldsymbol{V} \tag{50}$$

where β is a coefficient involving both shape and dimension of the satellite. Here the value $\beta = 5.75\text{D–03}\ \text{m}^2/\text{kg}$ was adopted, which corresponds to a spherical satellite, ρ is the atmospheric density assumed to be constant and equal to $\rho = 0.2\text{D–10}\ \text{kg/m}^3$, and V is the modulus of the velocity $\boldsymbol{V}$ of the satellite relative to the atmosphere.

If minutes and the mass and equatorial radius of the Earth (R_e = 6378 kms) are adopted as the units of time, mass, and distance, respectively, the gravitational parameter of the problem is given by $k = 0.07436574 R_e^{3/2}/\text{min}$. $T = 94.617$ min corresponds roughly to one revolution of the satellite and was defined arbitrarily as the revolution time.

The results of the application of the universal method of orders $p = 4$ and 5, with $w = 1$ min being the stepsize used, were compared to the solution given by Everhart's[12] integrator with a much smaller step length, namely, $w = 0.1$ min.

Thus, Table 4 shows the agreement achieved after one revolution of the satellite with a step size an order of magnitude larger than that used in Everhart's application. It should be stressed that 13 digits of coincidence in position were achieved by the universal method of order 4, whose overall speed of computation was about 13 times higher than that of Everhart's integrator.

Table 5 diplays results similar to those of Table 4 but after 50 revolutions of the satellite.

The gain of precision by increasing the order p of the universal method from $p = 4$ to $p = 5$ is about four digits, but the computing time is barely increased.

This confirms what has been previously stated: It is advisable to achieve a required precision by increasing the number of c_k functions taken up in the series rather than by shortening the step length.

VIII. Conclusions

The universal scheme is applicable to systems that are dominated by a central mass, such as planetary or satellite systems, even if nonconservative forces are involved.

The universal formulation involves an analytical step regulation since the physical time has been replaced by the regularized time according to Sundman's differential law. The regularized equations of motion are solved by expanding position and velocity in telescoped Taylor series in the regularized time, telescoping being accomplished by the introduction of Stumpff's functions c_k.

In light of this interpretation, both numerical stability and convergence have been demonstrated. Moreover, error estimates have been derived and corroborated by numerical experiments.

The universal scheme with all of its advantages has been extended to cover gravitational N-body problems. The efficiency of the method has been tested for several solar-system configurations, including nonconservative dynamical systems, and it may be concluded that it compares favorably to conventional direct numerical integration. This is because the universal scheme takes full advantage of the integrability of the Keplerian problem. Further, in contrast to classical methods, it is advisable to achieve a required precision not by shortening the step length but rather by increasing the number of c_k functions taken up in the universal series.

In a certain way the universal scheme constitutes an automatic application of Encke's method,[13] tailored to the problem at hand; this is indeed its essential advantage.

Appendix A: Summary of Formulas for the Universal Solution of the Perturbed Two-Body Problem

Equations of Motion

The vectorial equation of motion of the perturbed body is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r} + k^2(M+m)\frac{\boldsymbol{r}}{r^3} = \mathbb{P} \tag{A1}$$

with the initial conditions

$$\boldsymbol{r}_{(t_0)} = \boldsymbol{r}_0 \qquad \dot{\boldsymbol{r}}_{(t_0)} = \dot{\boldsymbol{r}}_0$$

with $\mathbb{P}$ a given perturbing acceleration and $\boldsymbol{r}_0$ and $\dot{\boldsymbol{r}}_0$ given vectors of initial position and velocity. The only requirement about the perturbing function $\mathbb{P}$ is to have a clear definition of it to be able to calculate its value and that of its successive time derivatives either analytically or numerically. Thus, the universal formulation can be applied to problems involving nonpoint mass potentials, nonconservative perturbations, and so forth. If the perturbation is a result of the attraction of a third body of mass m_p and position vector $\boldsymbol{R}$,

$$\mathbb{P} = k^2 m_p \left\{ \frac{\boldsymbol{R} - \boldsymbol{r}}{|\boldsymbol{R} - \boldsymbol{r}|^3} - \frac{\boldsymbol{R}}{|\boldsymbol{R}|^3} \right\} \tag{A2}$$

Introducing a new function of time

$$\tau = k\sqrt{M + m}(t - t_0) \tag{A3}$$

Eq. (A1) reduces to the simpler form

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\boldsymbol{r} + \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} = \boldsymbol{P} \tag{A4}$$

where

$$\boldsymbol{P} = \frac{\mathbb{P}}{(M + m)} \tag{A5}$$

In heliocentric or geocentric motions $M = 1$ is usually adopted and if the perturbed body is an asteroid, comet, or artificial body, m is considered as negligible.

Local Invariables

The local invariables are defined by

$$\mu_0 = \frac{1}{(\boldsymbol{r}_0 \cdot \boldsymbol{r}_0)^{\frac{3}{2}}} \tag{A6}$$

$$\sigma_0 = \frac{(\boldsymbol{r}_0 \cdot \dot{\boldsymbol{r}}_0)}{(\boldsymbol{r}_0 \cdot \boldsymbol{r}_0)} \tag{A7}$$

$$\omega_0 = \frac{(\dot{\boldsymbol{r}}_0 \cdot \dot{\boldsymbol{r}}_0)}{(\boldsymbol{r}_0 \cdot \boldsymbol{r}_0)} \tag{A8}$$

$$\varepsilon_0 = \omega_0 - \mu_0 \tag{A9}$$

$$\rho_0 = \mu_0 - \varepsilon_0 \tag{A10}$$

from which are calculated

$$\xi_0 = \mu_0 \tau^2 \tag{A11}$$

$$\eta_0 = \sigma_0 \tau \tag{A12}$$

$$\zeta_0 = \varepsilon_0 \tau^2 \tag{A13}$$

$$\chi_0 = \rho_0 \tau^2 \tag{A14}$$

with τ given by Eq. (A3).

Main Equation

The value of the universal variable z corresponding to a requested time t results from the main equation

$$z + c_2 \eta_0 z^2 + c_3 \zeta_0 z^3 = 1 - \Phi_{(z)} \tag{A15}$$

where

$$\Phi = \tau^2 z^3 \left(c_3 a_1 + c_4 a_2 \tau z + c_5 a_3 \tau^2 z^2 + c_6 a_4 \tau^3 z^3 + \cdots \right) \tag{A16}$$

which is to be solved by a process of successive iterations; a convenient first approximation is $z = 1$.

The coefficients a_k are calculated by

$$a_1 = \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0^2} \tag{A17}$$

$$a_2 = \frac{3(\dot{\boldsymbol{r}}_0 \cdot \boldsymbol{P}_0) + (\boldsymbol{r}_0 \cdot \dot{\boldsymbol{P}}_0)}{r_0^2}$$

$$a_3 = 3\sigma_0 a_2 - 3\mu_0 a_1 + b$$

$$a_4 = \left\{ 4[\varepsilon_0 + a_1] + 3\sigma_0^2 - \mu_0 \right\} a_2 + 6\sigma_0 b + c - 3\mu_0 \left\{ 3\sigma_0 a_1 + \frac{2(\boldsymbol{r}_0 \cdot \dot{\boldsymbol{P}}_0)}{r_0^2} \right\}$$

$$a_5 = d + 10\sigma_0 c + 15\sigma_0^2 b + 5a_2^2 + 5(\varepsilon_0 + a_1)(a_3 + b) - \mu_0\{a_3 + 6[a_1(\varepsilon_0 + a_1) + \sigma_0 a_2]\} - \left(\mu_0 / r_0^2\right)\{11(\boldsymbol{r}_0 \ddot{\boldsymbol{P}}_0)\} + 6(\dot{\boldsymbol{r}}_0 \dot{\boldsymbol{P}}_0) + 36\sigma_0(\boldsymbol{r}_0 \dot{\boldsymbol{P}}_0)\}$$

where

$$b = \frac{3(\boldsymbol{P}_0 \cdot \boldsymbol{P}_0) + 4(\dot{\boldsymbol{r}}_0 \cdot \dot{\boldsymbol{P}}_0) + (\boldsymbol{r}_0 \cdot \ddot{\boldsymbol{P}}_0)}{r_0^2}$$

$$c = \frac{10(\boldsymbol{P}_0 \cdot \dot{\boldsymbol{P}}_0) + 5(\dot{\boldsymbol{r}}_0 \cdot \ddot{\boldsymbol{P}}_0) + (\boldsymbol{r}_0 \cdot \dddot{\boldsymbol{P}}_0)}{r_0^2}$$

$$d = \frac{15(\boldsymbol{P}_0 \cdot \ddot{\boldsymbol{P}}_0) + 10(\dot{\boldsymbol{P}}_0 \cdot \dot{\boldsymbol{P}}_0) + 6(\dot{\boldsymbol{r}}_0 \cdot \dddot{\boldsymbol{P}}_0) + (\boldsymbol{r}_0 \cdot \ddddot{\boldsymbol{P}}_0)}{r_0^2}$$

and c_k are the universal functions

$$c_{k(\lambda^2)} = \sum_{i=0}^{\infty} \frac{(-\lambda^2)^i}{(2i + k)!} \tag{A18}$$

which satisfy the recurrence relationship

$$1/k! = c_k + \lambda^2 c_{k+2} \tag{A19}$$

The argument of the universal functions to be adopted is

$$\lambda^2 = \left[\chi_0 - \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0^2} \tau^2 \right] z^2 \tag{A20}$$

Position and Velocity at Instant t

The position and velocity vectors of the perturbed body for any instant t are obtained in the form

$$\boldsymbol{r} = F\boldsymbol{r}_0 + G\dot{\boldsymbol{r}}_0 + H\boldsymbol{g}_0 \tag{A21}$$

$$\dot{\boldsymbol{r}} = \dot{F}\boldsymbol{r}_0 + \dot{G}\dot{\boldsymbol{r}}_0 + \dot{H}\boldsymbol{g}_0 \tag{A22}$$

where $\boldsymbol{g}_0 = \boldsymbol{r}_0 \times \dot{\boldsymbol{r}}_0$ is the angular momentum vector, orthogonal to the instantaneous plane of motion of the perturbed body at the instant t_0, and F, G, H, $\dot{F}$, $\dot{G}$, and $\dot{H}$ are scalar functions given in terms of z by

$$F_{(z)} = 1 - c_2 \xi_0 z^2 + \tau^2 z^2 \left\{ c_2 f_1 + c_3 f_2 \tau z + c_4 f_3 \tau^2 z^2 + c_5 f_4 \tau^3 z^3 + \cdots \right\}$$

$$G_{(z)} = \tau z\{c_1 - c_2 \eta_0 z\} + \tau^2 z^2 \left\{ c_2 g_1 + c_3 g_2 \tau z + c_4 g_3 \tau^2 z^2 + c_5 g_4 \tau^3 z^3 + \cdots \right\}$$

$$H_{(z)} = \tau^2 z^2 \left\{ c_2 h_1 + c_3 h_2 \tau z + c_4 h_3 \tau^2 z^2 + c_5 h_4 \tau^3 z^3 + \cdots \right\} \tag{A23}$$

and

$$\dot{F}_{(z)} = -c_1(\mu_0 \tau z / \Delta) + (\tau z / \Delta) \left\{ c_1 f_1 + c_2 f_2 \tau z + c_3 f_3 \tau^2 z^2 + c_4 f_4 \tau^3 z^3 + c_5 f_5 \tau^4 z^4 + \cdots \right\}$$

$$\dot{G}_{(z)} = \frac{\{c_0 + c_1 n_0 z\}}{\Delta} + \frac{\tau z}{\Delta} \left\{ c_1 g_1 + c_2 g_2 \tau z + c_3 g_3 \tau^2 z^2 + c_4 g_4 \tau^3 z^3 + c_5 g_5 \tau^4 z^4 + \cdots \right\}$$

$$\dot{H}_{(z)} = (\tau z / \Delta) \left\{ c_1 h_1 + c_2 h_2 \tau z + c_3 h_3 \tau^2 z^2 + c_4 h_4 \tau^3 z^3 + c_5 h_5 \tau^4 z^4 + \cdots \right\} \tag{A24}$$

where

$$\Delta = r/r_0 = 1 + c_1 n_0 z + c_2 \zeta_0 z^2 + \Phi^*_{(z)} \qquad \text{(A25)}$$

with

$$\Phi^* = \tau^2 z^2 \left(c_2 a_1 + c_3 a_2 \tau z + c_4 a_3 \tau^2 z^2 + c_5 a_4 \tau^3 z^3 + \cdots\right) \qquad \text{(A26)}$$

where c_k are the universal functions of argument (A20) and a_k is given by Eq. (A17). The coefficients f_k, g_k, and h_k depend on the local invariables and on the components of $\boldsymbol{P}$ and its time derivatives along the directions defined by the position, velocity, and angular momentum vectors at t_0. The first coefficients can be calculated according to

$$\begin{pmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \\ f_4 & g_4 & h_4 \\ f_5 & g_5 & h_5 \end{pmatrix} = \mathbb{A} \begin{pmatrix} P_{r_0} & P_{V_0} & P_{g_0} \\ \dot{P}_{r_0} & \dot{P}_{V_0} & \dot{P}_{g_0} \\ \ddot{P}_{r_0} & \ddot{P}_{V_0} & \ddot{P}_{g_0} \\ \dddot{P}_{r_0} & \dddot{P}_{V_0} & \dddot{P}_{g_0} \\ \ddddot{P}_{r_0} & \ddddot{P}_{V_0} & \ddddot{P}_{g_0} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_2 & 0 \\ -2\mu_0 a_2 & a_3 & 0 \\ 3\mu_0(2\sigma_0 a_2 - a_3) & a_4 - 3\mu_0 a_2 & 0 \end{pmatrix} \qquad \text{(A27)}$$

where the 5×5 matrix $\mathbb{A}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3\sigma_0 & 1 & 0 & 0 & 0 \\ 3\left(\varepsilon_0 + a_1 + \sigma_0^2\right) & 6\sigma_0 & 1 & 0 & 0 \\ 5a_2 + 3\sigma_0[4(\varepsilon_0 + a_1) - \mu_0] & 9(\varepsilon_0 + a_1) + 15\sigma_0^2 & 10\sigma_0 & 1 & 0 \\ \mathbb{A}_{51} & \mathbb{A}_{52} & 19(\varepsilon_0 + a_1) + 45\sigma_0^2 & 15\sigma_0 & 1 \end{pmatrix} \qquad \text{(A28)}$$

with

$$\mathbb{A}_{51} = 6a_3 + 3\sigma_0(5a_2 - 4\mu_0\sigma_0) + 3(\varepsilon_0 + a_1) \times \left[4\left(\varepsilon_0 + a_1 + \sigma_0^2\right) - \mu_0\right]$$

$$\mathbb{A}_{52} = 15a_2 + 3\sigma_0\left[23(\varepsilon_0 + a_1) - 4\mu_0 + 5\sigma_0^2\right]$$

Appendix B: Partial Derivatives in the State Transition Matrix

Expressions for the partial derivatives $\delta F/\delta x_k$, $\delta G/\delta x_k$, $\delta H/\delta x_k$, $\delta \dot{F}/\delta x_k$, $\delta \dot{G}/\delta x_k$, $\delta \dot{H}/\delta x_k$, $k = 1 \to 6$, valid to the first order, are given in terms of the regularized time q.

For the sake of brevity, we define the trascendental functions

$$S_k = q^k c_k\left(\alpha_0^2 q^2\right) \qquad \text{(B1)}$$

where c_k denotes the universal function of argument $\lambda^2 = \alpha_0^2 q^2$.

In this notation the expressions obtained can be conveniently summarized as follows:

$$\frac{\partial F}{\partial x_0} = \left\{\mathbb{F}_1 + \frac{2}{r_0^3}\mathbb{F}_2 + 2S_2 P_{r_0} - \dot{F} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\} \times x_0 + \mathbb{F}_3 \dot{x}_0 + \{\mathbb{F}_2 - \dot{F} S_3 r_0\} P_{x_0} \qquad \text{(B2)}$$

$$\frac{\partial F}{\partial y_0} = \left\{\mathbb{F}_1 + \frac{2}{r_0^3}\mathbb{F}_2 + 2S_2 P_{r_0} - \dot{F} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\} \times y_0 + \mathbb{F}_3 \dot{y}_0 + \{\mathbb{F}_2 - \dot{F} S_3 r_0\} P_{y_0} \qquad \text{(B3)}$$

$$\frac{\partial F}{\partial z_0} = \left\{\mathbb{F}_1 + \frac{2}{r_0^3}\mathbb{F}_2 + 2S_2 P_{r_0} - \dot{F} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\} \times z_0 + \mathbb{F}_3 \dot{z}_0 + \{\mathbb{F}_2 - \dot{F} S_3 r_0\} P_{z_0} \qquad \text{(B4)}$$

and

$$\frac{\partial F}{\partial \dot{x}_0} = \mathbb{F}_3 x_0 + 2\mathbb{F}_2 \dot{x}_0 \qquad \text{(B5)}$$

$$\frac{\partial F}{\partial \dot{y}_0} = \mathbb{F}_3 y_0 + 2\mathbb{F}_2 \dot{y}_0 \qquad \text{(B6)}$$

$$\frac{\partial F}{\partial \dot{z}_0} = \mathbb{F}_3 z_0 + 2\mathbb{F}_2 \dot{z}_0 \qquad \text{(B7)}$$

with $\mathbb{F}_1$, $\mathbb{F}_2$, and $\mathbb{F}_3$ given by

$$\mathbb{F}_1 = \left(S_2/r_0^2\right) - \dot{F}(S_1/r_0) \qquad \text{(B8)}$$

$$\mathbb{F}_2 = \left(1/\alpha_0^2\right)\{(F - 1) - \dot{F}\mathbb{A}\}$$

$$\mathbb{A} = (3\tau/2) - \left[r_0 S_1 + \tfrac{1}{2} r_0^2 \sigma_0 S_2\right] \qquad \text{(B9)}$$

$$\mathbb{F}_3 = -\dot{F} S_2 \qquad \text{(B10)}$$

$$\frac{\partial G}{\partial x_0} = \left\{\mathbb{G}_1 + \frac{2}{r_0^3}\mathbb{G}_2 + 2S_2 P_{V_0} - \dot{G} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\} \times x_0 + \mathbb{G}_3 \dot{x}_0 + \{\mathbb{G}_2 - \dot{G} S_3 r_0\} P_{x_0} \qquad \text{(B11)}$$

$$\frac{\partial G}{\partial y_0} = \left\{\mathbb{G}_1 + \frac{2}{r_0^3}\mathbb{G}_2 + 2S_2 P_{V_0} - \dot{G} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\} \times y_0 + \mathbb{G}_3 \dot{y}_0 + \{\mathbb{G}_2 - \dot{G} S_3 r_0\} P_{y_0} \qquad \text{(B12)}$$

$$\frac{\partial G}{\partial z_0} = \left\{\mathbb{G}_1 + \frac{2}{r_0^3}\mathbb{G}_2 + 2S_2 P_{V_0} - \dot{G} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\} \times z_0 + \mathbb{G}_3 \dot{z}_0 + \{\mathbb{G}_2 - \dot{G} S_3 r_0\} P_{z_0} \qquad \text{(B13)}$$

and

$$\frac{\partial G}{\partial \dot{x}_0} = \mathbb{G}_3 x_0 + 2\mathbb{G}_2 \dot{x}_0 \qquad \text{(B14)}$$

$$\frac{\partial G}{\partial \dot{y}_0} = \mathbb{G}_3 y_0 + 2\mathbb{G}_2 \dot{y}_0 \qquad \text{(B15)}$$

$$\frac{\partial G}{\partial \dot{z}_0} = \mathbb{G}_3 z_0 + 2\mathbb{G}_2 \dot{z}_0 \qquad \text{(B16)}$$

where

$$\mathbb{G}_1 = (1 - \dot{G})(S_1/r_0) \qquad \text{(B17)}$$

$$\mathbb{G}_2 = \frac{1}{\alpha_0^2}\left\{G - \dot{G}\mathbb{A} - \frac{r_0}{2}S_1 + \frac{1}{2}\sum_{k=2}^{\infty}(k-1) g_k S_{k+1}\right\} \qquad \text{(B18)}$$

$$\mathbb{G}_3 = (1 - \dot{G}) S_2 \qquad \text{(B19)}$$

Finally,

$$\frac{\partial H}{\partial x_0} = \left\{\mathbb{H}_1 + \frac{2}{r_0^3}\mathbb{H}_2 + 2S_2 P_{g0} - \dot{H} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\}$$

$$\times x_0 + \mathbb{H}_3 \dot{x}_0 + \{\mathbb{H}_2 - \dot{H} S_3 r_0\} P_{x0} \tag{B20}$$

$$\frac{\partial H}{\partial y_0} = \left\{\mathbb{H}_1 + \frac{2}{r_0^3}\mathbb{H}_2 + 2S_2 P_{g0} - \dot{H} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\}$$

$$\times y_0 + \mathbb{H}_3 \dot{y}_0 + \{\mathbb{H}_2 - \dot{H} S_3 r_0\} P_{y0} \tag{B21}$$

$$\frac{\partial H}{\partial z_0} = \left\{\mathbb{H}_1 + \frac{2}{r_0^3}\mathbb{H}_2 + 2S_2 P_{g0} - \dot{H} S_3 \frac{(\boldsymbol{r}_0 \cdot \boldsymbol{P}_0)}{r_0}\right\}$$

$$\times z_0 + \mathbb{H}_3 \dot{z}_0 + \{\mathbb{H}_2 - \dot{H} S_3 r_0\} P_{z0} \tag{B22}$$

and

$$\frac{\partial H}{\partial \dot{x}_0} = \mathbb{H}_3 x_0 + 2\mathbb{H}_2 \dot{x}_0 \tag{B23}$$

$$\frac{\partial H}{\partial \dot{y}_0} = \mathbb{H}_3 y_0 + 2\mathbb{H}_2 \dot{y}_0 \tag{B24}$$

$$\frac{\partial H}{\partial \dot{z}_0} = \mathbb{H}_3 z_0 + 2\mathbb{H}_2 \dot{z}_0 \tag{B25}$$

where

$$\mathbb{H}_1 = -\dot{H}(S_1/r_0) \tag{B26}$$

$$\mathbb{H}_2 = \frac{1}{\alpha_0^2}\left\{H - \dot{H}\mathbb{A} + \frac{1}{2}\sum_{k=2}^{\infty}(k-1)h_k S_{k+1}\right\} \tag{B27}$$

$$\mathbb{H}_3 = -\dot{H} S_2 \tag{B28}$$

Obviously all of these partial derivatives, obtained by differentiating the solution for coordinates and manipulating the results algebraically, reduce for $\boldsymbol{P} \equiv 0$ to those given by Sconzo[14] for the unperturbed Keplerian motion.

Expressions for the partial derivatives $\partial \dot{F}/\partial x_k, \partial \dot{G}/\partial x_k, \partial \dot{H}/\partial x_k$, are obtained as an inmediate consequence of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial x_k}\right) = \frac{\partial \dot{F}}{\partial x_k} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G}{\partial x_k}\right) = \frac{\partial \dot{G}}{\partial x_k}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial x_k}\right) = \frac{\partial \dot{H}}{\partial x_k} \tag{B29}$$

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